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(Non)local Hamiltonian and symplectic structures, recursions and hierarchies: a new approach and applications to the $N = 1$ supersymmetric KdV equation

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Abstract
Using methods of Kersten *et al* (2004 *J. Geom. Phys.* **50** 273–302) and Krasil′shchik and Kersten (2000 *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations* (Dordrecht: Kluwer)), we accomplish an extensive study of the $N = 1$ supersymmetric Korteweg–de Vries (KdV) equation. The results include a description of local and nonlocal Hamiltonian and symplectic structures, five hierarchies of symmetries, the corresponding hierarchies of conservation laws, recursion operators for symmetries and generating functions of conservation laws. We stress that the main point of the paper is not just the results on super-KdV equation itself, but merely exposition of the efficiency of the geometrical approach and of the computational algorithms based on it.

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Introduction

There exist a number of superextensions of the classical KdV equation

$$u_t = -u_{xxx} + 6uu_x$$

(see [15] and the references therein). One of them, the so-called $N = 1$ supersymmetric extension, is

$$u_t = -u_{xxx} + 6uu_x + \varphi_{xx}\varphi \qquad \varphi_t = -\varphi_{xxx} + 3u\varphi_x + 3u_x\varphi \tag{1}$$

where φ is an odd (fermionic) variable [17]. To deal with this system, it is convenient to introduce a new independent odd variable θ such that $D_\theta^2 = D_x$, where

$$D_\theta = \partial_\theta + \theta D_x$$

(here D_x denotes the total derivative operator; see below) and a new odd function

$$\Phi = \varphi + \theta u.$$

Then (1) will acquire the form

$$\Phi_t = -\Phi_{xxx} + 3D_\theta(\Phi)\Phi_x + 3D_\theta(\Phi_x)\Phi. \tag{2}$$

This equation is linear in θ and reduces to (1) if we equal to each other the corresponding coefficients at the left- and right-hand sides. System (1) (or equation (2)) was studied before (see, e.g., [13]), and a number of results related to its integrability were obtained. The aim of our paper is twofold: (1) to represent the known results in a more convenient form (at least, from our point of view); (2) to demonstrate the efficiency of new methods of analysis of integrable systems described in [7, 8] and based on a general geometric approach to nonlinear PDE [2, 10]. Actually, description of *these methods and their highly algorithmical nature* (and, to a less extent, of the results on the super-KdV equation themselves) is the main goal of the paper. For traditional approach to the Hamiltonian formalism in integrable systems, we refer the reader to [3, 12, 18, 20]; an extensive exposition of the theory for superintegrable systems can be found in [11].

This paper is organized as follows. In section 1, we present the essential definitions and results needed for applications paying main attention to the computational aspects rather than to theoretical ones. All the proofs can be found in [2, 7, 8, 10]. In section 2, the results for the $N = 1$ supersymmetric KdV equation are described. Finally, in the last section we briefly discuss the results and perspectives.

1. Description of the computational scheme

Here we deal with evolution systems $\mathcal{E}$ of the form

$$v_t = F(y, t, v_1, \ldots, v_k) \tag{3}$$

where both the unknown variables $v = (v^1, \ldots, v^m)$ and the right-hand side $F = (F^1, \ldots, F^m)$ are vector-functions and $v_i = \partial^i v/\partial y^i$, y and t being the independent variables.

Remark 1. In applications, some of the variables v^j, as well as y, may be *odd*. In particular, in equation (2) θ and Φ are odd and x is even. Nevertheless, for the sake of simplicity, we expose the general theory for purely *even* equations. Necessary corrections are needed for the super case and the reader will find them in subsection 1.10.

Two basic operators related to (3),

$$D_y = \frac{\partial}{\partial y} + \sum_{i,j} v_{i+1}^j \frac{\partial}{\partial v_i^j} \qquad D_t = \frac{\partial}{\partial t} + \sum_{i,j} D_y^i(F^j)\frac{\partial}{\partial v_i^j}$$

are called the *total derivatives*.

Remark 2. Note that the above expressions for total derivatives contain infinite number of terms. To make the action of these operators (as well as of similar operators introduced below) well defined, we introduce the space $\mathcal{F}(\mathcal{E})$ of functions smoothly depending on y, t and a *finite number* of variables v_i^j, and assume D_y and D_t to act in this space. Similarly, we shall consider the spaces $\mathcal{F}^m(\mathcal{E})$ of vector-functions of length m that depend on y, t and v_i^j in the same way.

1.1. Symmetries

A *symmetry* of equation (3) is a vector field

$$S = \sum_{i,j} S_i^j \frac{\partial}{\partial v_i^j} \qquad S_i^j \in \mathcal{F}(\mathcal{E})$$

such that

$$[S, D_y] = [S, D_t] = 0.$$

Any symmetry is of the form

$$\partial_f = \sum_{i,j} D_y^i(f^j) \frac{\partial}{\partial v_i^j} \tag{4}$$

where the vector-function $f = (f^1, \ldots, f^m) \in \mathcal{F}^m(\mathcal{E})$ satisfies the system of equations

$$D_t(f^l) = \sum_{i,j} \frac{\partial F^l}{\partial v_i^j} D_y^i(f^j) \qquad l = 1, \ldots, m. \tag{5}$$

The operator at the right-hand side of (5) is called the *linearization* of F and is denoted by ℓ_F. Thus, equation (5) acquires the form

$$D_t(f) = \ell_F(f). \tag{6}$$

There exists a one-to-one correspondence between symmetries (4) and the corresponding functions $f \in \mathcal{F}^m(\mathcal{E})$; hence, we shall identify symmetries with such functions and use the term 'symmetry' for any function that satisfies (6).

1.2. Conservation laws and generating functions

A *conservation law* of system (3) is a pair $\Omega = (Y, T)$, $Y, T \in \mathcal{F}(\mathcal{E})$, such that

$$D_t(Y) = D_y(T). \tag{7}$$

The function Y is called the *density* of Ω. A conservation law is called *trivial* if $Y = D_y(P)$, $T = D_t(P)$ for some function $P \in \mathcal{F}(\mathcal{E})$.

To any conservation law there corresponds its *generating function* defined by

$$g_\Omega = \frac{\delta Y}{\delta v} = \left(\frac{\delta Y}{\delta v^1}, \ldots, \frac{\delta Y}{\delta v^m}\right)$$

where

$$\frac{\delta}{\delta v^j} = \sum_{i \geqslant 0} (-D_y)^i \circ \frac{\partial}{\partial v_i^j}$$

is the *variational derivative* with respect to v^j. Generating functions of conservation laws satisfy the system of equations

$$D_t(g) = -\ell_F^*(g) \tag{8}$$

or

$$D_t(g^l) = -\sum_{i,j} (-D_y)^i \left(\frac{\partial F^j}{\partial v_i^l} g^j\right) \qquad l = 1, \ldots, m \tag{9}$$

where ℓ_F^* is *adjoint* to the operator ℓ_F.

Any conservation law is uniquely determined by its generating function and, in particular, Ω is trivial if and only if $g_\Omega = 0$. We stress that equation (9) may possess solutions that do not correspond to any conservation law of (3).

Remark 3. Generating functions are also called *cosymmetries* [1] or *conserved covariants* [4].

1.3. Nonlocal variables

Let us introduce a set of variables $w^1, \dots, w^j, \dots$ satisfying the equations

$$w^j_y = A^j\big(y, t, \dots, v^\alpha_i, \dots, w^\beta, \dots\big) \qquad w^j_t = B^j\big(y, t, \dots, v^\alpha_i, \dots, w^\beta, \dots\big) \tag{10}$$

that are compatible modulo equation (3), where A^j, B^j are some smooth functions depending on a finite number of arguments. Consider the operators

$$\widetilde{D}_y = D_y + \sum_j A^j \frac{\partial}{\partial w^j} \qquad \widetilde{D}_t = D_t + \sum_j B^j \frac{\partial}{\partial w^j}.$$

Due to compatibility conditions, one has

$$[\widetilde{D}_y, \widetilde{D}_t] = 0 \tag{11}$$

modulo (3). Variables w^j are called *nonlocal*.

Using the operators $\widetilde{D}_y, \widetilde{D}_t$ instead of D_y and D_t in formulae (5), (7) and (9), we can introduce the notions of *nonlocal symmetries*, *nonlocal conservation laws* and *nonlocal generating functions* depending on the new variables w^j. We shall denote the spaces of such symmetries and generating functions by $\mathbf{sym}(\mathcal{E})$ and $\mathbf{gf}(\mathcal{E})$, respectively.

Remark 4. An invariant geometric way to introduce nonlocal variables is based on the notion of *covering*, see [2, 8–10].

1.4. The ℓ- and ℓ^*-extensions

There are two canonical ways to extend the initial system (3). The first one is related to the operator ℓ_F and is called the *ℓ-extension*. Namely, let us introduce the nonlocal variables ω^j_i (we shall also denote ω^j_0 by ω^j), $j = 1, \dots, m$, $i = 0, 1, \dots$, satisfying the relations

$$\big(\omega^j_i\big)_y = \omega^j_{i+1} \qquad \big(\omega^j_i\big)_t = \widetilde{D}^i_y \left(\sum_{s,l} \frac{\partial F^j}{\partial v^l_s}\omega^l_s\right).$$

Clearly, these equations are consistent modulo (3) and are the consequences of the following ones:

$$\omega^j_t = \sum_{i,l} \frac{\partial F^j}{\partial v^l_i}\omega^l_i. \tag{12}$$

In a similar way, we construct the *ℓ^*-extension*: the nonlocal variables are p^j_i (p^j_0 will also be denoted by p^j), and the defining relations are

$$\big(p^j_i\big)_y = p^j_{i+1} \qquad \big(p^j_i\big)_t = -\widetilde{D}^i_y \left(\sum_{s,l} (-\widetilde{D}_y)^s \left(\frac{\partial F^l}{\partial v^j_s} p^l\right)\right)$$

that reduce to the equations

$$p^j_t = -\sum_{s,l} (-\widetilde{D}_y)^s \left(\frac{\partial F^l}{\partial v^j_s} p^l\right) \tag{13}$$

and their differential consequences.

Remark 5. The parities of the variables ω^j and p^j are opposite to that of v^j: if v^j is *even*, then ω^j and p^j are *odd* and vice versa.

If the initial equation $\mathcal{E}$ was extended by nonlocal variables w^j, we can associate with these variables, in a canonical way, the corresponding ω and p whose 'behaviour' is governed by linearization or, respectively, adjoint linearization of equations (10) in the corresponding nonlocal setting.

1.4.1. Associating operators to functions on the ℓ- and ℓ^*-extensions. Let $\mathcal{F}^m(\mathcal{E})$ be the space of vector-valued functions of length m (see remark 2). Consider the case when $\mathcal{E}$ is not extended by nonlocal variables first. Let $a = (a_1, \dots, a_m)$, $a_i = \sum_{jl} a_l^{ij} \omega_l^j$, $a_l^{ij} \in \mathcal{F}(\mathcal{E})$, be a linear in ω vector function. Then we put into correspondence to this function a differential operator $\Delta_a = \|\Delta_a^{ij}\| \colon \mathcal{F}^m(\mathcal{E}) \to \mathcal{F}^m(\mathcal{E})$, where

$$\Delta_a^{ij} = \sum_l a_l^{ij} D_y^l \qquad i, j = 1, \dots, m.$$

If $\mathcal{F}(\mathcal{E})$ contains nonlocal variables, the situation becomes more complicated. We shall consider here the simplest case when the functions A^j in (10) are independent of ω^β. Let $\bar{\omega}^\beta$ be a variable in the ℓ-extension associated with the nonlocal variable w^β and $b = (b^1, \dots, b^m)$, $b^i = \sum_\beta b^{i\beta} \bar{\omega}^\beta$, be a linear in $\bar{\omega}$ vector-function. Then the corresponding operator $\Delta_b = \|\Delta_b^{ij}\| \colon \mathcal{F}^m(\mathcal{E}) \to \mathcal{F}^m(\mathcal{E})$ is of the form

$$\Delta_b^{ij} = \sum_\alpha b^{i\alpha} D_y^{-1} \circ \sum_l \frac{\partial A^\alpha}{\partial v_l^j} D_y^l. \tag{14}$$

For the ℓ^*-extension the construction is completely similar.

Below we shall use the notation $\mathcal{L}^m(\ell_\mathcal{E})$ and $\mathcal{L}^m(\ell_\mathcal{E}^*)$ for the spaces of vector functions linear in ω, $\bar{\omega}$ and p, $\bar{p}$, respectively.

1.5. Recursion operators for symmetries

Let $R \in \mathcal{L}^m(\ell_\mathcal{E})$ be a function that satisfies the equation

$$\widetilde{D}_t(R) = \widetilde{\ell}_F(R).$$

Then the corresponding operator Δ_R maps $\mathbf{sym}(\mathcal{E})$ to $\mathbf{sym}(\mathcal{E})$ and thus is a recursion operator for (nonlocal) symmetries of $\mathcal{E}$.

Remark 6. Here and below $\widetilde{\ell}_F$ denotes the linearization operator with the total derivative D_y replaced by its counterpart $\widetilde{D}_y$ for the ℓ- or ℓ^*-covering, and $\widetilde{\ell}_F^*$ stands for the adjoint of $\widetilde{\ell}_F$.

1.6. Recursion operators for generating functions

Let $L \in \mathcal{L}^m(\ell_\mathcal{E}^*)$ be a function that satisfies the equation

$$\widetilde{D}_t(L) = -\widetilde{\ell}_F^*(L).$$

Then the corresponding operator Δ_L maps $\mathbf{gf}(\mathcal{E})$ to $\mathbf{gf}(\mathcal{E})$ and thus is a recursion operator for (nonlocal) generating functions of $\mathcal{E}$ (or *adjoint recursion operator* [1]).

1.7. Hamiltonian structures

Let $K \in \mathcal{L}^m(\ell_\mathcal{E}^*)$ be a function that satisfies the equation

$$\widetilde{D}_t(K) = \widetilde{\ell}_F(K).$$

Then the corresponding operator Δ_K maps $\mathbf{gf}(\mathcal{E})$ to $\mathbf{sym}(\mathcal{E})$. We call such maps *pre-Hamiltonian structures* (they are also known as *Noether operators* [4]). In order to Δ_K be a true *Hamiltonian structure*, it has to satisfy two conditions: skew-symmetry ($\Delta_K^* = -\Delta_K$) and the Jacobi identity for the corresponding Poisson bracket (that amounts to $[\![\Delta_K, \Delta_K]\!] = 0$, where $[\![\cdot, \cdot]\!]$ is the *variational Schouten bracket*, see [5, 7]). Both of these conditions are easily checked in terms of the function K.

Namely, if $K = \left\|\sum_{jl} a_l^{ij} p_l^j\right\|$ then we consider the function $W_K = \sum_{ijl} a_l^{ij} p_l^j p^i$ and in terms of W_K, the first condition reads

$$\sum_i \frac{\delta W_K}{\delta p^i} p^i = -2W_K \tag{15}$$

while the second one is

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta p}\right) \sum_i \left(\frac{\delta W_K}{\delta v^i} \frac{\delta W_K}{\delta p^i}\right) = 0 \tag{16}$$

$(\delta/\delta v, \delta/\delta p) = (\delta/\delta v^1, \ldots, \delta/\delta v^m, \delta/\delta p^1, \ldots, \delta/\delta p^m)$. Note also that the compatibility condition for two Hamiltonian structures K and K' amounts to

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta p}\right) \sum_i \left(\frac{\delta W_K}{\delta v^i} \frac{\delta W_{K'}}{\delta p^i} + \frac{\delta W_{K'}}{\delta v^i} \frac{\delta W_K}{\delta p^i}\right) = 0. \tag{17}$$

The equation $\mathcal{E}$ itself is in the Hamiltonian form if it possesses a Hamiltonian structure K and may be presented as

$$v_t = \Delta_K \frac{\delta Y}{\delta v} \tag{18}$$

for some function Y.

1.8. Symplectic structures

Let $J \in \mathcal{L}^m(\ell_{\mathcal{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(J) = -\widetilde{\ell}_F^*(J).$$

Then the corresponding operator Δ_J, which maps $\mathbf{sym}(\mathcal{E})$ to $\mathbf{gf}(\mathcal{E})$, is called an *inverse Noether operator* [4] for $\mathcal{E}$. An operator $\Delta_J : \mathbf{sym}(\mathcal{E}) \to \mathbf{gf}(\mathcal{E})$, not necessary being an inverse Noether operator, is called *symplectic* (or a *symplectic structure*) cf e.g., [4, 14], if it enjoys the following properties. Let $J = \left\|\sum_{jl} b_l^{ij} \omega_l^j\right\|$. Similar to subsection 1.7, we consider the function $W_J = \sum_{ijl} b_l^{ij} \omega_l^j \omega^i$ and impose the conditions

$$\sum_i \frac{\delta W_J}{\delta \omega^i} \omega^i = -2W_J \tag{19}$$

i.e. the operator Δ_J is skew-adjoint, and

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta \omega}\right) \sum_i \frac{\delta W_J}{\delta v^i} \omega^i = 0 \tag{20}$$

that means that the 'form' W_J is closed. Thus, in our context, the term 'symplectic structure' means the same as in classical mechanics, cf [14].

1.9. Canonical representation

As it will be seen below, all the operators constructed in our study are presented in the form

$$\sum_{\alpha \geqslant 0} c_{ij}^\alpha D_y^\alpha + \sum_\beta d_j^\beta D_y^{-1} \circ e_i^\beta$$

where $\left\|c_{ij}^\alpha\right\|$ is an $m \times m$-matrix, $\left\|d_j^\beta\right\|$ is an $m \times l$-matrix and $\left\|e_i^\beta\right\|$ is an $l \times m$-matrix for some $l > 0$ (matrix-valued functions, to be more precise). In the table it is shown how the matrices d and e look for different types of operators.

Type of operator	Lines of matrix d	Columns of matrix e
Recursions for symmetries	Symmetry	Generating function
Recursions for generating function	Generating function	Symmetry
Hamiltonian structures	Symmetry	Symmetry
Symplectic structures	Generating function	Generating function

1.10. Super case

We shall now assume that all objects under consideration belong to the supersetting, i.e. may be either even or odd, which means that they obey the rule

$$AB = (-1)^{AB} BA.$$

Here and below, symbols used at the exponents of (-1) stand for the corresponding parity. Generalization of the above exposed theory to the super case is carried out along the lines of [10, 19].

Then the basic formulae to be used in the calculus described above are

(1) For evolutionary derivations

$$\mathcal{D}_\varphi = \sum_{ij} (-1)^{\varphi v_i^j} D_y^i(\varphi^j) \frac{\partial}{\partial v_i^j}$$

(naturally, the parity of v_i^j equals that of v^j plus parity of y times i).

(2) For the linearization one has $\ell_f(\varphi) = (-1)^{f\varphi} \mathcal{D}_\varphi(f)$ that amounts to

$$(\ell_f)_\alpha^\beta = \sum_i (-1)^{(f^\alpha+1)v_i^\beta} \frac{\partial f^\alpha}{\partial v_i^\beta} D_y^i.$$

(3) For the operator adjoint to $\Delta = \sum_i a_i D_y^i$ one has

$$\Delta^* = \sum_i (-1)^{i + i a_i y + \frac{i(i-1)}{2} y} D_y^i \circ a_i.$$

2. Main results for the $N = 1$ supersymmetric KdV equation

Here we apply the theory described above to equation (2)

$$\Phi_t = -\Phi_{xxx} + 3D_\theta(\Phi)\Phi_x + 3D_\theta(\Phi_x)\Phi.$$

We use the notation

$$\Phi_k \qquad \text{for} \quad \frac{\partial^{2k}\Phi}{\partial\theta^{2k}} = \frac{\partial^k \Phi}{\partial x^k}$$

and

$$\Phi_{k\frac{1}{2}} \qquad \text{for} \quad D_\theta^{2k+1}(\Phi) = D_\theta\left(\frac{\partial^k\Phi}{\partial x^k}\right).$$

The functions Φ_k are *odd* while $\Phi_{k\frac{1}{2}}$ are *even*; the function $\Phi = \Phi_0$ itself being odd.

Gradings. We assign the following gradings $[\cdot]$ to the variables on our equation:

$$[\theta] = -\tfrac{1}{2} \qquad [x] = -1 \qquad [t] = -3 \qquad [\Phi] = \tfrac{3}{2}$$

respectively, we have

$$[\Phi_k] = (2k+3)/2 \qquad \left[\Phi_{k\frac{1}{2}}\right] = k+2.$$

With these gradings, equation (2) becomes homogeneous (of grading $9/2$) and all constructions below can be considered to be homogeneous as well.

2.1. Nonlocal functions

Here we extend the equation $\mathcal{E}$ by four groups of nonlocal variables. We present here their θ-components only; the x- and t-components are given in [6] (they are found from compatibility conditions (11)).

2.1.1. Group 1.

This group includes the even variables q_1, q_3, q_5, defined by

$$(q_1)_\theta = \Phi_0$$
$$(q_3)_\theta = \Phi_0\Phi_{\frac{1}{2}}$$
$$(q_5)_\theta = \Phi_{\frac{1}{2}}\big(-\Phi_2 + 2\Phi_0\Phi_{\frac{1}{2}}\big)\big/2.$$

Gradings: $[q_1] = 1, [q_3] = 3, [q_5] = 5$.

2.1.2. Group 2.

This group includes the odd variables $Q_{\frac{1}{2}}, Q_{\frac{5}{2}}, Q_{\frac{9}{2}}$ defined by

$$\big(Q_{\frac{1}{2}}\big)_\theta = q_1$$
$$\big(Q_{\frac{5}{2}}\big)_\theta = q_1^3 - 6q_3$$
$$\big(Q_{\frac{9}{2}}\big)_\theta = -60\Phi_0\Phi_1 q_1 + q_1^5 - 60q_1^2 q_3 + 240q_5.$$

Gradings: $\big[Q_{\frac{1}{2}}\big] = \frac{1}{2}, \big[Q_{\frac{5}{2}}\big] = \frac{5}{2}, \big[Q_{\frac{9}{2}}\big] = \frac{9}{2}$.

2.1.3. Group 3.

This group includes the odd variables $Q_{\frac{3}{2}}, Q_{\frac{7}{2}}, Q_{\frac{11}{2}}$ defined by

$$\big(Q_{\frac{3}{2}}\big)_\theta = \Phi_0 Q_{\frac{1}{2}}$$
$$\big(Q_{\frac{7}{2}}\big)_\theta = \big(12\Phi_2 Q_{\frac{1}{2}} + 18\Phi_1 Q_{\frac{1}{2}} q_1 + \Phi_0 Q_{\frac{5}{2}}\big)\big/3$$
$$\big(Q_{\frac{11}{2}}\big)_\theta = \big(360\Phi_4 Q_{\frac{1}{2}} + 5280\Phi_3 Q_{\frac{1}{2}} q_1 - 760\Phi_2 Q_{\frac{5}{2}} + 4680\Phi_2 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 1200\Phi_2 Q_{\frac{1}{2}} q_1^2 + 60\Phi_1 Q_{\frac{5}{2}} q_1 + \Phi_0 Q_{\frac{9}{2}}\big)\big/60.$$

Gradings: $\big[Q_{\frac{3}{2}}\big] = \frac{3}{2}, \big[Q_{\frac{7}{2}}\big] = \frac{7}{2}, \big[Q_{\frac{11}{2}}\big] = \frac{11}{2}$.

2.1.4. Group 4.

This group includes the even variables $\bar{q}_1, \bar{q}_3, \bar{q}_5$ defined by

$$(\bar{q}_1)_\theta = Q_{\frac{3}{2}}$$
$$(\bar{q}_3)_\theta = -\big(Q_{\frac{7}{2}} + Q_{\frac{3}{2}} q_1^2\big)$$
$$(\bar{q}_5)_\theta = \big(12Q_{\frac{11}{2}} + 42Q_{\frac{7}{2}}\Phi_{\frac{1}{2}} + 6Q_{\frac{7}{2}} q_1^2 + 12Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}} q_1 + Q_{\frac{3}{2}} q_1^4 - 24Q_{\frac{3}{2}} q_1 q_3\big)\big/3.$$

Gradings: $[\bar{q}_1] = 1, [\bar{q}_3] = 3, [\bar{q}_5] = 5$.

Remark 7. The last three variables are not used directly in the subsequent computations, but clarify the nonlocal picture and enter in the expressions for the higher terms of hierarchies of symmetries and generating functions.

2.2. Seed symmetries

Solving equation (5), which in our case is of the form

$$\widetilde{D}_t(f) = -\widetilde{D}_\theta^6(f) + 3\widetilde{D}_\theta(f)\Phi_1 + 3\Phi_{\frac{1}{2}}\widetilde{D}_\theta^2(f) + 3\widetilde{D}_\theta^3(f)\Phi + 3\Phi_{1\frac{1}{2}} f$$

where $\widetilde{D}_\theta = \partial_\theta + \theta\widetilde{D}_x$, while $\widetilde{D}_x$ and $\widetilde{D}_t$ are the total derivative operators extended to the nonlocal setting (see subsection 2.1), we found a number of solutions that serve as seed symmetries for constructing infinite hierarchies and are used to construct *nonlocal vectors* (see subsection 2.4).

These symmetries are

The Y_k series.

$$Y_1 = \Phi_1$$
$$Y_3 = \Phi_3 - 3\Phi_1\Phi_{\frac{1}{2}} - 3\Phi_0\Phi_{1\frac{1}{2}}$$
$$Y_5 = \Phi_5 - 5\Phi_3\Phi_{\frac{1}{2}} - 10\Phi_2\Phi_{1\frac{1}{2}} + 10\Phi_1\Phi_{\frac{1}{2}}^2 - 10\Phi_1\Phi_{2\frac{1}{2}} + 20\Phi_0\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - 5\Phi_0\Phi_{3\frac{1}{2}}.$$

The $Y_{k\frac{1}{2}}$ series.

$$Y_{\frac{3}{2}} = -2\Phi_1 Q_{\frac{1}{2}} - \Phi_{\frac{1}{2}}q_1 + \Phi_{1\frac{1}{2}}$$
$$\begin{aligned} Y_{\frac{7}{2}} = {} & -12\Phi_3 Q_{\frac{1}{2}} - 2\Phi_1 Q_{\frac{5}{2}} + 36\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 36\Phi_0 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 12\Phi_0\Phi_2 - 6\Phi_0\Phi_1 q_1 \\ & + 12\Phi_{\frac{1}{2}}^2 q_1 - 36\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - \Phi_{\frac{1}{2}}q_1^3 + 6\Phi_{\frac{1}{2}}q_3 + 3\Phi_{1\frac{1}{2}}q_1^2 - 6\Phi_{2\frac{1}{2}}q_1 + 6\Phi_{3\frac{1}{2}} \end{aligned}$$
$$\begin{aligned} Y_{\frac{11}{2}} = {} & 240\Phi_5 Q_{\frac{1}{2}} + 40\Phi_3 Q_{\frac{5}{2}} - 1200\Phi_3 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 2400\Phi_2 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} \\ & + 2\Phi_1 Q_{\frac{9}{2}} - 120\Phi_1 Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 2400\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 2400\Phi_1 Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 600\Phi_1\Phi_3 \\ & + 240\Phi_1\Phi_2 q_1 - 120\Phi_0 Q_{\frac{5}{2}}\Phi_{1\frac{1}{2}} + 4800\Phi_0 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - 1200\Phi_0 Q_{\frac{1}{2}}\Phi_{3\frac{1}{2}} \\ & - 480\Phi_0\Phi_4 + 360\Phi_0\Phi_3 q_1 + 1920\Phi_0\Phi_2\Phi_{\frac{1}{2}} - 120\Phi_0\Phi_2 q_1^2 - 720\Phi_0\Phi_1\Phi_{\frac{1}{2}}q_1 \\ & + 1680\Phi_0\Phi_1\Phi_{1\frac{1}{2}} + 20\Phi_0\Phi_1 q_1^3 - 120\Phi_0\Phi_1 q_3 + 660\Phi_{\frac{1}{2}}^3 q_1 - 3540\Phi_{\frac{1}{2}}^2\Phi_{1\frac{1}{2}} \\ & - 40\Phi_{\frac{1}{2}}^2 q_1^3 + 240\Phi_{\frac{1}{2}}^2 q_3 + 360\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}q_1^2 - 960\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}}q_1 + 1200\Phi_{\frac{1}{2}}\Phi_{3\frac{1}{2}} \\ & + \Phi_{\frac{1}{2}}q_1^5 - 60\Phi_{\frac{1}{2}}q_1^2 q_3 + 240\Phi_{\frac{1}{2}}q_5 - 720\Phi_{1\frac{1}{2}}^2 q_1 + 2400\Phi_{1\frac{1}{2}}\Phi_{2\frac{1}{2}} - 5\Phi_{1\frac{1}{2}}q_1^4 \\ & + 120\Phi_{1\frac{1}{2}}q_1 q_3 + 20\Phi_{2\frac{1}{2}}q_1^3 - 120\Phi_{2\frac{1}{2}}q_3 - 60\Phi_{3\frac{1}{2}}q_1^2 + 120\Phi_{4\frac{1}{2}}q_1 - 120\Phi_{5\frac{1}{2}}. \end{aligned}$$

The Z_k series.

$$Z_1 = Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + \theta\left(-2\Phi_1 Q_{\frac{1}{2}} - \Phi_{\frac{1}{2}}q_1 + \Phi_{1\frac{1}{2}}\right)$$
$$\begin{aligned} Z_3 = {} & \big(3Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}q_1 - 3Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}} + Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} - 12Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 3Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}q_1 + 6Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} \\ & + 6\Phi_1 Q_{\frac{1}{2}}Q_{\frac{3}{2}} + 6\Phi_0\Phi_1 Q_{\frac{1}{2}} + \theta\big(-12\Phi_3 Q_{\frac{1}{2}} - 2\Phi_1 Q_{\frac{5}{2}} + 36\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} \\ & + 36\Phi_0 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 12\Phi_0\Phi_2 - 6\Phi_0\Phi_1 q_1 + 12\Phi_{\frac{1}{2}}^2 q_1 - 36\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} \\ & - \Phi_{\frac{1}{2}}q_1^3 + 6\Phi_{\frac{1}{2}}q_3 + 3\Phi_{1\frac{1}{2}}q_1^2 - 6\Phi_{2\frac{1}{2}}q_1 + 6\Phi_{3\frac{1}{2}}\big)\big)/3 \end{aligned}$$
$$\begin{aligned} Z_5 = {} & \big(-15Q_{\frac{7}{2}}\Phi_{\frac{1}{2}}q_1 + 15Q_{\frac{7}{2}}\Phi_{1\frac{1}{2}} + 120Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}^2 q_1 - 360Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} \\ & - 10Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}q_1^3 + 60Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}q_3 + 30Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}}q_1^2 - 60Q_{\frac{3}{2}}\Phi_{2\frac{1}{2}}q_1 + 60Q_{\frac{3}{2}}\Phi_{3\frac{1}{2}} \\ & - Q_{\frac{9}{2}}\Phi_{\frac{1}{2}} + 40Q_{\frac{5}{2}}\Phi_{\frac{1}{2}}^2 - 5Q_{\frac{5}{2}}\Phi_{\frac{1}{2}}q_1^2 + 15Q_{\frac{5}{2}}\Phi_{1\frac{1}{2}}q_1 - 20Q_{\frac{5}{2}}\Phi_{2\frac{1}{2}} - 660Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^3 \\ & + 90Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 q_1^2 - 390Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}q_1 + 960Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}} + 5Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}q_1^4 \\ & - 30Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}q_1 q_3 + 660Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}^2 - 10Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}q_1^3 - 30Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}q_3 + 60Q_{\frac{1}{2}}\Phi_{3\frac{1}{2}}q_1 \\ & - 120Q_{\frac{1}{2}}\Phi_{4\frac{1}{2}} + 12\Phi_5 - 120\Phi_3 Q_{\frac{1}{2}}Q_{\frac{3}{2}} - 60\Phi_3\Phi_{\frac{1}{2}} - 120\Phi_2\Phi_{1\frac{1}{2}} \\ & - 20\Phi_1 Q_{\frac{5}{2}}Q_{\frac{3}{2}} - 30\Phi_1 Q_{\frac{1}{2}}Q_{\frac{7}{2}} + 360\Phi_1 Q_{\frac{1}{2}}Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - 10\Phi_1 Q_{\frac{1}{2}}Q_{\frac{5}{2}}q_1 \\ & - 240\Phi_1\Phi_2 Q_{\frac{1}{2}} - 60\Phi_1\Phi_{\frac{1}{2}}q_1^2 + 60\Phi_1\Phi_{1\frac{1}{2}}q_1 - 120\Phi_1\Phi_{2\frac{1}{2}} + 360\Phi_0 Q_{\frac{1}{2}}Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}} \end{aligned}$$

$$- 360\Phi_0\Phi_3 Q_{\frac12} + 120\Phi_0\Phi_2 Q_{\frac32} + 120\Phi_0\Phi_2 Q_{\frac12} q_1 + 60\Phi_0\Phi_1 Q_{\frac32} q_1$$
$$- 20\Phi_0\Phi_1 Q_{\frac52} + 720\Phi_0\Phi_1 Q_{\frac12}\Phi_{\frac12} - 180\Phi_0\Phi_1 Q_{\frac12} q_1^2 + 300\Phi_0\Phi_{\frac12}\Phi_{1\frac12}$$
$$- 90\Phi_0\Phi_{\frac12} q_1^3 + 90\Phi_0\Phi_{1\frac12} q_1^2 - 60\Phi_0\Phi_{3\frac12} + \theta\big(240\Phi_5 Q_{\frac12} + 40\Phi_3 Q_{\frac52}$$
$$- 1200\Phi_3 Q_{\frac12}\Phi_{\frac12} - 2400\Phi_2 Q_{\frac12}\Phi_{1\frac12} + 2\Phi_1 Q_{\frac92} - 120\Phi_1 Q_{\frac52}\Phi_{\frac12}$$
$$+ 2400\Phi_1 Q_{\frac12}\Phi_{\frac12}^2 - 2400\Phi_1 Q_{\frac12}\Phi_{2\frac12} - 600\Phi_1\Phi_3 + 240\Phi_1\Phi_2 q_1$$
$$- 120\Phi_0 Q_{\frac52}\Phi_{1\frac12} + 4800\Phi_0 Q_{\frac12}\Phi_{\frac12}\Phi_{1\frac12} - 1200\Phi_0 Q_{\frac12}\Phi_{3\frac12} - 480\Phi_0\Phi_4$$
$$+ 360\Phi_0\Phi_3 q_1 + 1920\Phi_0\Phi_2\Phi_{\frac12} - 120\Phi_0\Phi_2 q_1^2 - 720\Phi_0\Phi_1\Phi_{\frac12} q_1$$
$$+ 1680\Phi_0\Phi_1\Phi_{1\frac12} + 20\Phi_0\Phi_1 q_1^3 - 120\Phi_0\Phi_1 q_3 + 660\Phi_{\frac12}^3 q_1 - 3540\Phi_{\frac12}^2\Phi_{1\frac12}$$
$$- 40\Phi_{\frac12}^2 q_1^3 + 240\Phi_{\frac12}^2 q_3 + 360\Phi_{\frac12}\Phi_{1\frac12} q_1^2 - 960\Phi_{\frac12}\Phi_{2\frac12} q_1 + 1200\Phi_{\frac12}\Phi_{3\frac12}$$
$$+ \Phi_{\frac12} q_1^5 - 60\Phi_{\frac12} q_1^2 q_3 + 240\Phi_{\frac12} q_5 - 720\Phi_{1\frac12}^2 q_1 + 2400\Phi_{1\frac12}\Phi_{2\frac12}$$
$$- 5\Phi_{1\frac12} q_1^4 + 120\Phi_{1\frac12} q_1 q_3 + 20\Phi_{2\frac12} q_1^3 - 120\Phi_{2\frac12} q_3 - 60\Phi_{3\frac12} q_1^2 + 120\Phi_{4\frac12} q_1$$
$$- 120\Phi_{5\frac12}\big)\big)/5.$$

The $Z_{k\frac12}$ series.

$$Z_{\frac12} = -2\theta\Phi_1 + \Phi_{\frac12}$$
$$Z_{\frac52} = -2\Phi_1 Q_{\frac32} + \Phi_1 Q_{\frac12} q_1 + 2\Phi_0\Phi_1 - 4\Phi_{\frac12}^2 + \Phi_{\frac12} q_1^2 - 2\Phi_{1\frac12} q_1 + 2\Phi_{2\frac12}$$
$$+ \theta\big(-4\Phi_3 + 12\Phi_1\Phi_{\frac12} + 12\Phi_0\Phi_{1\frac12}\big)$$
$$Z_{\frac92} = -24\Phi_3 Q_{\frac32} + 24\Phi_3 Q_{\frac12} q_1 - 6\Phi_1 Q_{\frac72} + 72\Phi_1 Q_{\frac32}\Phi_{\frac12} + 2\Phi_1 Q_{\frac52} q_1$$
$$- 36\Phi_1 Q_{\frac12}\Phi_{\frac12} q_1 + 24\Phi_1 Q_{\frac12}\Phi_{1\frac12} - 36\Phi_1 Q_{\frac12} q_3 + 48\Phi_1\Phi_2 + 72\Phi_0 Q_{\frac32}\Phi_{1\frac12}$$
$$- 72\Phi_0 Q_{\frac12}\Phi_{1\frac12} q_1 + 72\Phi_0\Phi_3 - 48\Phi_0\Phi_2 q_1 - 144\Phi_0\Phi_1\Phi_{\frac12} + 48\Phi_0\Phi_1 q_1^2$$
$$+ \theta\big(-48\Phi_5 + 240\Phi_3\Phi_{\frac12} + 480\Phi_2\Phi_{1\frac12} - 480\Phi_1\Phi_{\frac12}^2 + 480\Phi_1\Phi_{2\frac12}$$
$$- 960\Phi_0\Phi_{\frac12}\Phi_{1\frac12} + 240\Phi_0\Phi_{3\frac12}\big) + 132\Phi_{\frac12}^3 - 24\Phi_{\frac12}^2 q_1^2 + 144\Phi_{\frac12}\Phi_{1\frac12} q_1$$
$$- 192\Phi_{\frac12}\Phi_{2\frac12} + \Phi_{\frac12} q_1^4 - 24\Phi_{\frac12} q_1 q_3 - 144\Phi_{1\frac12}^2 - 4\Phi_{1\frac12} q_1^3 + 24\Phi_{1\frac12} q_3$$
$$+ 12\Phi_{2\frac12} q_1^2 - 24\Phi_{3\frac12} q_1 + 24\Phi_{4\frac12}.$$

Gradings. There are two points of view on symmetries: as on functions and as on vector fields $\ni_f$ (see subsection 1.1). For functions, we have

$[Y_1] = \frac52$	$[Y_3] = \frac92$	$[Y_5] = \frac{13}{2}$	odd
$[Y_{\frac32}] = 3$	$[Y_{\frac72}] = 5$	$[Y_{\frac{11}{2}}] = 7$	even
$[Z_1] = \frac52$	$[Z_3] = \frac72$	$[Z_5] = \frac{13}{2}$	odd
$[Z_{\frac12}] = 2$	$[Z_{\frac52}] = 4$	$[Z_{\frac92}] = 6$	even.

For vector fields we have

$[\ni_{Y_1}] = 1$	$[\ni_{Y_3}] = 3$	$[\ni_{Y_5}] = 5$	even
$[\ni_{Y_{\frac32}}] = \frac32$	$[\ni_{Y_{\frac72}}] = \frac72$	$[\ni_{Y_{\frac{11}{2}}}] = \frac{11}{2}$	odd
$[\ni_{Z_1}] = 1$	$[\ni_{Z_3}] = 3$	$[\ni_{Z_5}] = 5$	even
$[\ni_{Z_{\frac12}}] = \frac12$	$[\ni_{Z_{\frac52}}] = \frac52$	$[\ni_{Z_{\frac92}}] = \frac92$	odd.

Note also that the symmetries Y_α do not depend on θ, while Z_α are linear functions with respect to θ.

2.3. Seed generating functions

Solving equation (5), which in our case is of the form

$$\widetilde{D}_t(f) = -\widetilde{D}_\theta^6(f) + 6\Phi_{\frac{1}{2}}\widetilde{D}_\theta^2(f) - 3\Phi_0\widetilde{D}_\theta^3(f)$$

we found a number of solutions that serve as seed generating functions for constructing infinite hierarchies and used to construct *nonlocal forms* (see subsection 2.5). These generating functions are

The F_k series.

$$F_0 = 1$$
$$F_2 = \Phi_{\frac{1}{2}}$$
$$F_4 = \left(-2\Phi_0\Phi_1 + 3\Phi_{\frac{1}{2}}^2 - \Phi_{2\frac{1}{2}}\right)/3.$$

The $F_{k\frac{1}{2}}$ series.

$$F_{\frac{1}{2}} = Q_{\frac{1}{2}}$$
$$F_{\frac{5}{2}} = \left(Q_{\frac{5}{2}} - 12Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 6\Phi_1 + 6\Phi_0 q_1\right)/6$$
$$F_{\frac{9}{2}} = \left(Q_{\frac{9}{2}} - 40Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 720Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 240Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} + 120\Phi_3 + 120\Phi_2 q_1 - 480\Phi_1\Phi_{\frac{1}{2}} + 60\Phi_1 q_1^2 - 480\Phi_0\Phi_1 Q_{\frac{1}{2}} - 420\Phi_0\Phi_{\frac{1}{2}}q_1 - 240\Phi_0\Phi_{1\frac{1}{2}} + 20\Phi_0 q_1^3 - 120\Phi_0 q_3\right)/20.$$

The G_k series.

$$G_0 = \theta Q_{\frac{1}{2}}$$
$$G_2 = \left(3Q_{\frac{1}{2}}Q_{\frac{3}{2}} + 6\Phi_0 Q_{\frac{1}{2}} + \theta Q_{\frac{5}{2}} - 12\theta Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 6\theta\Phi_1 + 6\theta\Phi_0 q_1\right)/3$$
$$G_4 = \left(-10Q_{\frac{5}{2}}Q_{\frac{3}{2}} + 15Q_{\frac{1}{2}}Q_{\frac{7}{2}} + 120Q_{\frac{1}{2}}Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - 5Q_{\frac{1}{2}}Q_{\frac{5}{2}}q_1 - 120\Phi_2 Q_{\frac{1}{2}} - 60\Phi_1 Q_{\frac{3}{2}} - 60\Phi_0 Q_{\frac{3}{2}}q_1 - 20\Phi_0 Q_{\frac{5}{2}} + 420\Phi_0 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 90\Phi_0 Q_{\frac{1}{2}}q_1^2 - 120\Phi_0\Phi_1 - \theta Q_{\frac{9}{2}} + 40\theta Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} - 720\theta Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 + 240\theta Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 120\theta\Phi_3 - 120\theta\Phi_2 q_1 + 480\theta\Phi_1\Phi_{\frac{1}{2}} - 60\theta\Phi_1 q_1^2 + 480\theta\Phi_0\Phi_1 Q_{\frac{1}{2}} + 420\theta\Phi_0\Phi_{\frac{1}{2}}q_1 + 240\theta\Phi_0\Phi_{1\frac{1}{2}} - 20\theta\Phi_0 q_1^3 + 120\theta\Phi_0 q_3\right)/90.$$

The $G_{k\frac{1}{2}}$ series.

$$G_{-\frac{1}{2}} = \theta$$
$$G_{\frac{3}{2}} = -Q_{\frac{3}{2}} + Q_{\frac{1}{2}}q_1 + 2\Phi_0 - 4\theta\Phi_{\frac{1}{2}}$$
$$G_{\frac{7}{2}} = \left(3Q_{\frac{7}{2}} - 24Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - Q_{\frac{5}{2}}q_1 + 6Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}q_1 - 12Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 18Q_{\frac{1}{2}}q_3 - 24\Phi_2 - 12\Phi_1 q_1 + 84\Phi_0\Phi_{\frac{1}{2}} + 6\Phi_0 q_1^2 + 96\theta\Phi_0\Phi_1 - 144\theta\Phi_{\frac{1}{2}}^2 + 48\theta\Phi_{2\frac{1}{2}}\right)/6.$$

Gradings. These generating functions have the following gradings and parities:

$[F_0] = 0$	$[F_2] = 2$	$[F_4] = 4$	even
$\left[F_{\frac{1}{2}}\right] = \frac{1}{2}$	$\left[F_{\frac{5}{2}}\right] = \frac{5}{2}$	$\left[F_{\frac{9}{2}}\right] = \frac{9}{2}$	odd
$[G_0] = 0$	$[G_2] = 2$	$[G_4] = 4$	even
$\left[G_{-\frac{1}{2}}\right] = -\frac{1}{2}$	$\left[G_{\frac{3}{2}}\right] = \frac{3}{2}$	$\left[G_{\frac{7}{2}}\right] = \frac{7}{2}$	odd.

Note again that the generating functions F_α do not depend on θ, while G_α are linear functions with respect to θ.

2.4. Nonlocal vectors

We consider now to the ℓ^*-extension of equation (2). The additional coordinates on this extension are denoted by $P = P_0, P_{\frac{1}{2}}, P_1$ etc.

Now we introduce nonlocal variables in the ℓ^*-extension that we call *nonlocal vectors* and which are defined by

$$\begin{array}{lll} \left(P_{Y_1}\right)_\theta = Y_1 P_0 & \left(P_{Y_3}\right)_\theta = Y_3 P_0 & \left(P_{Y_5}\right)_\theta = Y_5 P_0 \\ \left(P_{Y_{\frac{3}{2}}}\right)_\theta = Y_{\frac{3}{2}} P_0 & \left(P_{Y_{\frac{7}{2}}}\right)_\theta = Y_{\frac{7}{2}} P_0 & \left(P_{Y_{\frac{11}{2}}}\right)_\theta = Y_{\frac{11}{2}} P_0 \\ \left(P_{Z_1}\right)_\theta = Z_1 P_0 & \left(P_{Z_3}\right)_\theta = Z_3 P_0 & \left(P_{Z_5}\right)_\theta = Z_5 P_0 \\ \left(P_{Z_{\frac{1}{2}}}\right)_\theta = Z_{\frac{1}{2}} P_0 & \left(P_{Z_{\frac{5}{2}}}\right)_\theta = Z_{\frac{5}{2}} P_0 & \left(P_{Z_{\frac{9}{2}}}\right)_\theta = Z_{\frac{9}{2}} P_0 \end{array}$$

where the symmetries Y_α and Z_α were described in subsection 2.2.

The x- and t-components of these variables are given in [6].

Gradings. The variable P_0 is even and we assign grading 0 to it. Then P_k are also even variables with $[P_k] = k$, while $P_{k\frac{1}{2}}$ are odd and $\left[P_{k\frac{1}{2}}\right] = (2k+1)/2$. Consequently,

$$\begin{array}{llll} \left[P_{Y_1}\right] = 2 & \left[P_{Y_3}\right] = 4 & \left[P_{Y_5}\right] = 6 & \text{even} \\ \left[P_{Y_{\frac{3}{2}}}\right] = \frac{5}{2} & \left[P_{Y_{\frac{7}{2}}}\right] = \frac{9}{2} & \left[P_{Y_{\frac{11}{2}}}\right] = \frac{13}{2} & \text{odd} \\ \left[P_{Z_1}\right] = 2 & \left[P_{Z_3}\right] = 4 & \left[P_{Z_5}\right] = 6 & \text{even} \\ \left[P_{Z_{\frac{1}{2}}}\right] = \frac{3}{2} & \left[P_{Z_{\frac{5}{2}}}\right] = \frac{7}{2} & \left[P_{Z_{\frac{9}{2}}}\right] = \frac{11}{2} & \text{odd.} \end{array}$$

2.5. Nonlocal forms

Passing to the ℓ-extension of equation (2), we introduce the additional coordinates on this extension that are denoted by $\Omega = \Omega_0, \Omega_{\frac{1}{2}}, \Omega_1$, etc.

Now we introduce nonlocal variables in the ℓ-extension called *nonlocal forms* and described by

$$\begin{array}{lll} \left(\Omega_{F_0}\right)_\theta = \Omega_0 F_0 & \left(\Omega_{F_2}\right)_\theta = \Omega_0 F_2 & \left(\Omega_{F_4}\right)_\theta = \Omega_0 F_4 \\ \left(\Omega_{F_{\frac{1}{2}}}\right)_\theta = \Omega_0 F_{\frac{1}{2}} & \left(\Omega_{F_{\frac{5}{2}}}\right)_\theta = \Omega_0 F_{\frac{5}{2}} & \left(\Omega_{F_{\frac{9}{2}}}\right)_\theta = \Omega_0 F_{\frac{9}{2}} \\ \left(\Omega_{G_0}\right)_\theta = \Omega_0 G_0 & \left(\Omega_{G_2}\right)_\theta = \Omega_0 G_2 & \left(\Omega_{G_4}\right)_\theta = \Omega_0 G_4 \\ \left(\Omega_{G_{-\frac{1}{2}}}\right)_\theta = \Omega_0 G_{-\frac{1}{2}} & \left(\Omega_{G_{\frac{3}{2}}}\right)_\theta = \Omega_0 G_{\frac{3}{2}} & \left(\Omega_{G_{\frac{7}{2}}}\right)_\theta = \Omega_0 G_{\frac{7}{2}} \end{array}$$

where the generating functions F_α and G_α were described in subsection 2.3.

The x- and t-components of these variables are given in [6].

Gradings. The variable Ω_0 is even and we assign grading 0 to it. Then Ω_k are also even variables with $[\Omega_k] = k$, while $\Omega_{k\frac{1}{2}}$ are odd and $\left[\Omega_{k\frac{1}{2}}\right] = (2k+1)/2$. Consequently,

$$\begin{array}{llll} \left[\Omega_{F_0}\right] = -\frac{1}{2} & \left[\Omega_{F_2}\right] = \frac{3}{2} & \left[\Omega_{F_4}\right] = \frac{7}{2} & \text{odd} \\ \left[\Omega_{F_{\frac{1}{2}}}\right] = 0 & \left[\Omega_{F_{\frac{5}{2}}}\right] = 2 & \left[\Omega_{F_{\frac{9}{2}}}\right] = 4 & \text{even} \\ \left[\Omega_{G_0}\right] = -\frac{1}{2} & \left[\Omega_{G_2}\right] = \frac{3}{2} & \left[\Omega_{G_4}\right] = \frac{7}{2} & \text{odd} \\ \left[\Omega_{G_{-\frac{1}{2}}}\right] = -1 & \left[\Omega_{G_{\frac{3}{2}}}\right] = 1 & \left[\Omega_{G_{\frac{7}{2}}}\right] = 3 & \text{even.} \end{array}$$

2.6. Recursion operators for symmetries

Using the method described in subsection 1.5, we found two nontrivial solutions of the linearized equation in the ℓ-extension enriched with nonlocal variables. The first one is

$$R_1 = -Q_{\frac{1}{2}}\Omega_{F_0}\Phi_{\frac{1}{2}} - 2\Phi_1\Omega_{G_0} - \Phi_1\Omega_{F_0} + 2\Phi_1 Q_{\frac{1}{2}}\Omega_{G_{-\frac{1}{2}}} - 2\Phi_0\Omega_{\frac{1}{2}} + \theta\Omega_{F_0}\Phi_{\frac{1}{2}}q_1 - \theta\Omega_{F_0}\Phi_{1\frac{1}{2}} + 2\theta\Phi_1 Q_{\frac{1}{2}}\Omega_{F_0} + 2\theta\Phi_1\Omega_{F_{\frac{1}{2}}} - \Omega_{F_{\frac{1}{2}}}\Phi_{\frac{1}{2}} + \Omega_{G_{-\frac{1}{2}}}\Phi_{\frac{1}{2}}q_1 - \Omega_{G_{-\frac{1}{2}}}\Phi_{1\frac{1}{2}} - 2\Omega_0\Phi_{\frac{1}{2}} + \Omega_2.$$

The operator corresponding to the first solution is

$$\Delta_{R_1} = D_\theta^4 - 2\Phi_0 D_\theta - 2\Phi_{\frac{1}{2}} - (Y_1 + Z_1)D_\theta^{-1}\circ F_0 - Z_{\frac{1}{2}}D_\theta^{-1}\circ F_{\frac{1}{2}} - Y_{\frac{3}{2}}D_\theta^{-1}\circ G_{-\frac{1}{2}} - 2Y_1 D_\theta^{-1}\circ G_0.$$

This recursion operator coincides with the one found in [17]. The second solution is given in [6] and corresponds to the operator $\Delta_{R_1}^2$.

Gradings. The operator R_1 is even and its grading is 2.

2.7. Recursion operators for generating functions

Using the method described in subsection 1.6, we found three nontrivial solutions of the adjoint linearized equation in the ℓ^*-extension enriched with nonlocal variables. The first one is

$$L_1 = Q_{\frac{1}{2}}P_{Z_{\frac{1}{2}}} + 2\Phi_0 P_{\frac{1}{2}} + \theta P_{Y_{\frac{3}{2}}} + 2\theta Q_{\frac{1}{2}}P_{Y_1} - 4\Phi_{\frac{1}{2}}P_0 + P_{Y_1} + P_{Z_1} + P_2.$$

The operator corresponding to the first solution is

$$\Delta_{L_1} = D_\theta^4 + 2\Phi_0 D_\theta - 4\Phi_{\frac{1}{2}} + (F_0 + 2G_0)D_\theta^{-1}\circ Y_1 + G_{-\frac{1}{2}}D_\theta^{-1}\circ Y_{\frac{3}{2}} + F_0 D_\theta^{-1}\circ Z_1 + F_{\frac{1}{2}}D_\theta^{-1}\circ Z_{\frac{1}{2}}.$$

The second and third solutions are given in [6] and correspond to the operators $\Delta_{L_1}^2$ and $\Delta_{L_1}^3$, respectively.

Gradings. The operator L_1 is even and its grading is 2.

2.8. Hamiltonian structures

Using the method described in subsection 1.7, we found three nontrivial solutions of the linearized equation in the ℓ^*-extension enriched with nonlocal variables. The first one is

$$K_1 = P_{2\frac{1}{2}} - P_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 2\Phi_1 P_0 - 3\Phi_0 P_1.$$

The operator corresponding to the first solution is

$$\Delta_{K_1} = D_\theta^5 - 3\Phi_0 D_\theta^2 - \Phi_{\frac{1}{2}}D_\theta - 2\Phi_1.$$

This operator satisfies criteria (15) and (16) and thus is Hamiltonian. Moreover, there exists a conservation law (corresponding to the nonlocal variable q_3)

$$X = \Phi_0\Phi_{\frac{1}{2}}$$

$$T = -2\Phi_1\Phi_2 + \Phi_0\Phi_3 - 9\Phi_0\Phi_1\Phi_{\frac{1}{2}} + 4\Phi_{\frac{1}{2}}^3 - 2\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}} + \Phi_{1\frac{1}{2}}^2$$

such that our equation can be represented as

$$\Phi_t = \Delta_{K_1}\frac{\delta}{\delta\Phi}\left(-\frac{1}{2}X\right)$$

and so (18) is also satisfied. This Hamiltonian structure is well known, see, e.g., [15] and references therein.

The second Hamiltonian structure is of the form

$$K_2 = -P_{Z_{\frac12}}\Phi_{\frac12}q_1 + P_{Z_{\frac12}}\Phi_{1\frac12} - P_{Y_{\frac32}}\Phi_{\frac12} + P_{4\frac12} - 3P_{2\frac12}\Phi_{\frac12} - 3P_{1\frac12}\Phi_{1\frac12}$$
$$+3P_{\frac12}\Phi_{\frac12}^2 - P_{\frac12}\Phi_{2\frac12} - 2Q_{\frac12}\Phi_{\frac12}P_{Y_1} - 2\Phi_3P_0 - 7\Phi_2P_1 - 2\Phi_1Q_{\frac12}P_{Z_{\frac12}}$$
$$+9\Phi_1\Phi_{\frac12}P_0 - 2\Phi_1P_{Z_1} - 9\Phi_1P_2 - \Phi_0\Phi_1P_{\frac12} + 13\Phi_0\Phi_{\frac12}P_1 + 7\Phi_0\Phi_{1\frac12}P_0$$
$$-5\Phi_0P_3 + 2\theta\Phi_1P_{Y_{\frac32}} + 4\theta\Phi_1Q_{\frac12}P_{Y_1} + 2\theta\Phi_{\frac12}q_1P_{Y_1} - 2\theta\Phi_{1\frac12}P_{Y_1}.$$

The corresponding operator is

$$\Delta_{K_2} = D_\theta^9 - 5\Phi_0D_\theta^6 - 3\Phi_{\frac12}D_\theta^5 - 9\Phi_1D_\theta^4 - 3\Phi_{1\frac12}D_\theta^3 + \left(13\Phi_0\Phi_{\frac12} - 7\Phi_2\right)D_\theta^2$$
$$+\left(3\Phi_{\frac12}^2 - \Phi_{2\frac12} - \Phi_0\Phi_1\right)D_\theta + \left(9\Phi_1\Phi_{\frac12} + 7\Phi_0\Phi_{1\frac12} - 2\Phi_3\right)$$
$$+Y_{\frac32}D_\theta^{-1}\circ Z_{\frac12} - Z_{\frac12}D_\theta^{-1}\circ Y_{\frac32} - 2Y_1D_\theta^{-1}\circ Z_1 - 2Z_1D_\theta^{-1}\circ Y_1.$$

The third solution is given in [6] (see also remark 9).

Gradings. The operator Δ_{K_1} is odd and of grading $5/2$. The operator Δ_{K_2} is also odd and of grading $9/2$.

2.9. Symplectic structures

Using the method described in subsection 1.8, we found three nontrivial solutions of the adjoint linearized equation in the ℓ-extension enriched with nonlocal variables. The first one is

$$J_1 = \Omega_{G_0} + \Omega_{F_0} - Q_{\frac12}\Omega_{G_{-\frac12}} + \theta Q_{\frac12}\Omega_{F_0} + \theta\Omega_{F_{\frac12}}.$$

The operator corresponding to the first solution is

$$\Delta_{J_1} = (F_0 + G_0)D_\theta^{-1}\circ F_0 + G_{-\frac12}D_\theta^{-1}\circ F_{\frac12} - F_{\frac12}D_\theta^{-1}\circ G_{-\frac12} + F_0D_\theta^{-1}\circ G_0.$$

This operator can be shown to coincide with the known symplectic structure $D_\theta^{-1} - D_x^{-1}\circ\Phi\circ D_x^{-1}$, see [17] and also [15] and references therein.

The second solution is of the form

$$J_2 = \big(3\Omega_{G_2} - 12\Omega_{G_0}\Phi_{\frac12} - 12\Omega_{F_2} - 12\Omega_{F_0}\Phi_{\frac12} + 6\Omega_{1\frac12} - 3Q_{\frac32}\Omega_{F_{\frac12}}$$
$$-Q_{\frac52}\Omega_{G_{-\frac12}} + 3Q_{\frac12}Q_{\frac32}\Omega_{F_0} + 3Q_{\frac12}\Omega_{F_{\frac12}}q_1 + 12Q_{\frac12}\Omega_{G_{-\frac12}}\Phi_{\frac12}$$
$$-3Q_{\frac12}\Omega_{G_{\frac32}} - 6\Phi_1\Omega_{G_{-\frac12}} + 6\Phi_0Q_{\frac12}\Omega_{F_0} + 6\Phi_0\Omega_{F_{\frac12}} - 6\Phi_0\Omega_{G_{-\frac12}}q_1$$
$$+6\Phi_0\Omega_0 + \theta Q_{\frac52}\Omega_{F_0} - 12\theta Q_{\frac12}\Omega_{F_2} - 12\theta Q_{\frac12}\Omega_{F_0}\Phi_{\frac12} + 6\theta\Phi_1\Omega_{F_0}$$
$$+6\theta\Phi_0\Omega_{F_0}q_1 - 12\theta\Omega_{F_{\frac12}}\Phi_{\frac12} + 6\theta\Omega_{F_{\frac52}}\big)/6.$$

The corresponding symplectic structure is

$$\Delta_{J_2} = D_\theta^3 + \Phi_0 + \left(\tfrac12G_2 - 2F_2\right)D_\theta^{-1}\circ F_0$$
$$-2(F_0+G_0)D_\theta^{-1}\circ F_2 + \tfrac12G_{\frac32}D_\theta^{-1}\circ F_{\frac12} + G_{-\frac12}D_\theta^{-1}\circ F_{\frac52}$$
$$-2F_2D_\theta^{-1}\circ G_0 + \tfrac12F_0D_\theta^{-1}\circ G_2 - 6F_{\frac52}D_\theta^{-1}\circ G_{-\frac12} - \tfrac12F_{\frac12}D_\theta^{-1}\circ G_{\frac32}.$$

The third solution is given in [6] (see remark 9).

Gradings. The operator Δ_{J_1} is odd and of grading $-1/2$. The second operator is also odd and its grading equals $3/2$.

2.10. Interrelations

Using the symmetries computed in subsection 2.2 and applying the recursion operator obtained in subsection 2.6, we get four infinite series of (generally, nonlocal) symmetries

Y_{2k-1}	$[Y_{2k-1}] = (4k+1)/2$	odd
$Y_{\frac{4k-1}{2}}$	$\left[Y_{\frac{4k-1}{2}}\right] = 2k+1$	even
Z_{2k-1}	$[Z_{2k-1}] = (4k+1)/2$	odd
$Z_{\frac{4k-3}{2}}$	$\left[Z_{\frac{4k-3}{2}}\right] = 2k$	even

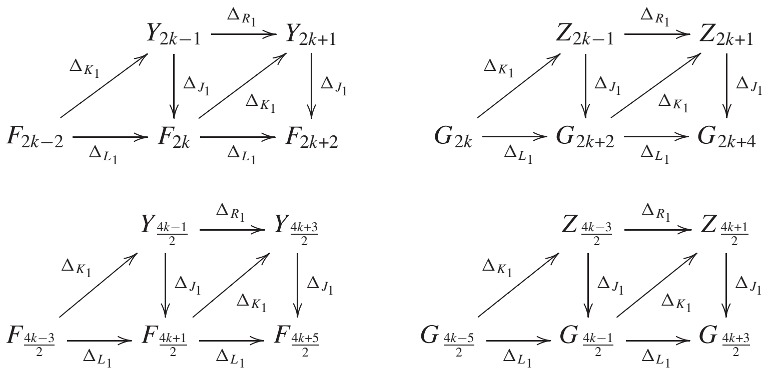
$k = 1, 2, \dots$.

In a similar war, using the results of subsections 2.3 and 2.7, we get four infinite series of generating functions

F_{2k-2}	$[F_{2k-2}] = 2k-2$	even
$F_{\frac{4k-3}{2}}$	$\left[F_{\frac{4k-3}{2}}\right] = (4k-3)/2$	odd
G_{2k}	$[G_{2k}] = 2k$	even
$G_{\frac{4k-5}{2}}$	$\left[G_{\frac{4k-5}{2}}\right] = (4k-5)/2$	odd

$k = 1, 2, \dots$.

These series are related to each other (up to rational coefficients) by the operators of subsections 2.6–2.9 in the following way:



Remark 8. Actually, there exists another hierarchy of symmetries S_{2k}, $k = 0, 1, \dots$, with the seed element

$$S_0 = 6\left(-\Phi_3 + 3\Phi_1\Phi_{\frac{1}{2}} + 3\Phi\Phi_{1\frac{1}{2}}\right)t + 2\Phi_1 x + \theta\Phi_{\frac{1}{2}} + 3\Phi$$

(the scaling symmetry). All these symmetries are odd, linear with respect to x, t, and θ, and have grading $[S_{2k}] = (4k+3)/2$, see [16] for the general properties of such hierarchies.

Remark 9 (cf [1])**.** Let us clarify the relations between the structures described above. First, it should be noted that the Hamiltonian structures K_1 and K_2 are *compatible*, i.e. their Schouten bracket vanishes (or, their linear combination is the Hamiltonian structure again). Moreover, they are related to each other by the recursion operator R_1: $\Delta_{K_2} = \Delta_{R_1} \circ \Delta_{K_1}$. Consequently, an infinite series of (nonlocal) compatible Hamiltonian structures K_i arises, such that $\Delta_{K_{i+1}} = \Delta_{R_1} \circ \Delta_{K_i}$. In a similar way, we have an infinite series of symplectic structures related by the operator L_1. In particular, $\Delta_{J_2} = \Delta_{L_1} \circ \Delta_{J_1}$. The inverse of each Hamiltonian structure, if it makes sense, is a symplectic structure and vice versa.

Second, in an obvious way all natural powers of recursion operators are also recursion operators. It is well known that if Δ_R is a recursion operator for symmetries, then its adjoint Δ_R^* is a recursion operator for generating functions, and vice versa. In particular, we have $\Delta_{R_1}^* = \Delta_{L_1}$.

3. Conclusion

The study of the $N = 1$ supersymmetric KdV equation exposed in this paper demonstrates the power and efficiency of the geometrical methods elaborated in [2] and [7]. In particular, we found recursion operators for symmetries and generating functions, Hamiltonian and symplectic structures and constructed five infinite series of symmetries. The research was based on new geometrical methods giving rise to efficient computational algorithms.

Our experience shows that the methods applied are of a universal nature and may be used to analyse a lot of other equations, both classical and supersymmetric. In particular, from technical point of view, the canonical representation of nonlocal operators (see subsection 1.9) seems to be quite efficient and convenient when dealing with such operators. Note that all nonlocal operators constructed in this paper are represented in the canonical form.

We strongly believe that the majority of the problems formulated in [15] can be solved by our methods. We plan to demonstrate this in forthcoming publications.

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